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Projections of Jordan bi-Poisson structures that are Kronecker, diagonal actions, and the classical Gaudin systems

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Abstract

We propose a method of constructing completely integrable systems based on reduction of bihamiltonian structures. More precisely, we give an easily checkable necessary and sufficient conditions for the micro-kroneckerity of the reduction (performed with respect to a special type action of a Lie group) of micro-Jordan bihamiltonian structures whose Nijenhuis tensor has constant eigenvalues. The method is applied to the diagonal action of a Lie group G on a direct product of N coadjoint orbits $\mathcal{O} = O_1 \times \cdots \times O_N \subset \mathfrak{g}^* \times \cdots \times \mathfrak{g}^*$ endowed with a bihamiltonian structure whose first generator is the standard symplectic form on $\mathcal{O}$. As a result we get the so-called classical Gaudin system on $\mathcal{O}$. The method works for a wide class of Lie algebras including the semisimple ones and for a large class of orbits including the generic ones and the semisimple ones.

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1. Introduction

In this paper we propose a method of constructing completely integrable systems, based on reduction of bihamiltonian structures. The method is illustrated by producing a class

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of systems on products of coadjoint orbits, which includes the so-called classical Gaudin systems. Now we will briefly explain the method and discuss the Gaudin systems.

According to the last decade investigations of bihamiltonian structures [8–10,23], i.e. pairs of compatible Poisson bivectors which will be called bi-Poisson structures in this paper, there are two main classes of them: micro-Jordan and micro-Kronecker (we shall omit the prefix micro- in this section for shortness). Jordan structures can be characterized by the property that almost every bivector in the corresponding pencil is nondegenerate, that is, the structure can be generated by the inverses of two symplectic forms $(\omega_1)^{-1}$, $(\omega_2)^{-1}$. Kronecker pencils consist of degenerate bivectors and are distinguished by the condition of the constancy of rank (see Section 3). Both classes of bi-Poisson structures play important role in completely integrable systems. Given a Jordan structure, one constructs an involutive family of functions by means of the corresponding Nijenhuis operator $N = \omega_2^{-1} \circ \omega_1$ (the eigenvalues of N are in involution and in various examples there are enough functionally independent ones); in Kronecker case functions in involution appear as Casimir functions of the Poisson bivectors of the pencil and form a complete set.

Among Jordan bi-Poisson structures there are ones which are trivial from the point of view of complete integrability: structures with the constant eigenvalues of N. We call them dull (after Zakharevich). It is amazing that using the simultaneous Poisson reduction of the symplectic forms ω_1, ω_2 related to such a structure one can produce a Kronecker bi-Poisson structure which is far from being “dull” since it gives a complete involutive family of functions, Casimirs of the Kronecker pencil. Due to this remark the following question seems to be important: when a simultaneous Poisson reduction of two symplectic forms generating a dull Jordan bi-Poisson structure gives a Kronecker one? We give necessary and sufficient conditions for such a reduction to be Kronecker in the situation which roughly can be described as follows: a Lie group G acts freely on a manifold M with a dull Jordan bi-Poisson structure; this action is hamiltonian with respect to all bivectors of the pencil; the induced actions on the spaces of symplectic leaves of the exceptional (i.e. of nonmaximal rank) bivectors are transitive (see Theorem 4.2).

Now, assume we are in such a situation and the necessary and sufficient conditions mentioned are satisfied. Then we are able to produce two complete involutive families of functions on M/G and M respectively. First of them, $\mathcal{F}$, is generated by all Casimir functions of bivectors from the constructed Kronecker pencil on M/G. It is involutive and complete with respect to any Poisson bivector of the pencil. The second one, $\mathcal{G}^{t_0}$, is related to any nondegenerate bivector η^{t_0} from the initial dull Jordan pencil $\{\eta^t = (\omega_1)^{-1} + t(\omega_2)^{-1} | t \in \mathbb{P}^1 = \mathbb{R}^1 \cup \infty\}$. Denoting by p the canonical projection $M \to M/G$, we define $\mathcal{G}^{t_0}$ as the family $p^*\mathcal{F}$ completed by $\mu_{t_0}^*\mathcal{F}'$, where μ_{t_0} is the corresponding moment map $M \to \mathfrak{g}^*$ and $\mathcal{F}'$ is a complete involutive set of functions on $\mathfrak{g}^*$ (endowed with the canonical linear Poisson bivector η_{can}). The family $\mathcal{G}^{t_0}$ is involutive and complete with respect to η^{t_0}.

Note, that due to the standard properties of the dual pairs of Poisson structures (cf. Section 2) the family $\mathcal{G}^{t_0}$ can be also generated by $\mu_{t_0}^*\mathcal{F}'$ and by $\{\mu_t^* Z^{\eta_{\mathrm{can}}} | t \in \mathbb{P}^1\}$ (instead of $p^*\mathcal{F}$), where $Z^{\eta_{\mathrm{can}}}$ is the set of Casimirs of η_{can}, i.e. invariants of coadjoint representation.

Next, we apply the method described above to the following data: $M = O_1 \times \cdots \times O_N \subset (\mathfrak{g}^*)^{\times N}$ is a coadjoint orbit of the Cartesian product $G^{\times N}$ of N copies of a Lie group G, G acting on M diagonally; $\omega_1 = \omega_{(1)} + \cdots + \omega_{(N)}$ is the standard symplectic form on M, $\omega_{(i)}$ being the standard symplectic form on O_i; ω_2 is defined as $(1/a_1)\omega_{(1)} + \cdots +$

$(1/a_N)\omega_{(N)}$, where $a_1, \dots, a_N$ are any different real numbers. Under some conditions on the orbits $O_1, \dots, O_N$ (see Theorem 5.3) Theorem 4.2 can be applied and we get a Kronecker bi-Poisson structure on the regular part of the variety M/G and the corresponding complete involutive sets of functions $\mathcal{F}$ and $\mathcal{G}^{t_0}$. By the remark above this last can be generated by $\mu_{t_0}^*\mathcal{F}'$ and $\{\mu_t^* Z^{\eta_{\mathrm{can}}} | t \in \mathbb{P}^1\}$, where $\mu_t(x_1, \dots, x_N) = (1/(t+a_1))x_1 + \cdots + (1/(t+a_N))x_N$, as calculations show. So, we recognize in $\mathcal{G}^{t_0}$ the so-called classical Gaudin integrable system.

The quantum version of this system, which uses the Lie algebra su(2) and describes some type of interaction of particles with spin, was introduced by Gaudin [5–7]. Later Sklyanin [19–21] studied separability of classical and quantum systems in case of $\mathfrak{g} = \mathrm{sl(n)}$ and with additional term in μ_t, a constant regular matrix. The integrability of such systems was also discussed in [18] from the point of view of r-matrix formalism.

Summarizing, the following items of this paper seem to be new: (1) the method of constructing completely integrable systems based on the reduction of dull Jordan bi-Poisson structures; (2) application of this method to the Gaudin type systems; (3) proof of the complete integrability of such systems for a wide class of Lie algebras including all semisimple ones and for a wide class of coadjoint orbits including all generic ones and all semisimple ones (see Section 5, in particular Remark 5.9).

The paper is organized as follows. Section 2 is preparatory: we introduce notations and recall standard definitions and facts related to Poisson structures, their dual pairs and hamiltonian actions. Proposition 2.21 is new (but easy). Proposition 2.22 is doubtlessly known but the author was not able to find its explicit formulation in the literature.

Similarly, Section 3 serves for introducing the notations and main data on bi-Poisson structures. The material of this section is more or less standard.

In Section 4 we give the first main result of this paper: necessary and sufficient conditions for the kroneckerity of the reduction of a dull Jordan bi-Poisson structure with respect to a specific bihamiltonian action of a Lie group (Theorem 4.2). In Corollary 4.3 we apply this result for constructing a completely integrable system on the initial manifold (the above mentioned family $\mathcal{G}^{t_0}$). We also illustrate the method by an example of a diagonal action of SL(2) on $\mathbb{R}^{2N}$ endowed with a dull Jordan bi-Poisson structure (see Example 4.4).

In Section 5 we develop this example and construct a dull Jordan bi-Poisson structure on a cartesian product of N coadjoint orbits of a Lie group G, whose reduction with respect to the diagonal action of G is Kronecker. Theorem 5.3 which establishes this kroneckerity using Theorem 4.2 is the second main result of the paper. Also, we calculate the corresponding families of the moment maps μ_t and complete involutive families of functions $\mathcal{F}$ and $\mathcal{G}^t$ (Corollaries 5.4–5.6). We conclude the paper by the discussion on the range of applicability of the method (Theorem 5.7, Lemma 5.8, Remarks 5.9 and 5.10 and Proposition 5.11).

2. Projections of Poisson structures, dual pairs and complete involutive sets of functions

Convention and Notations 2.1. All objects in this paper are real-analytic or complex analytic, M stands for a connected manifold, $\mathcal{E}(U)$ for a space of respectively real-valued analytic or holomorphic functions on an open set $U \subset M$. We shall write $\mathbb{K}$ for $\mathbb{R}$ or $\mathbb{C}$ depending on the category. The terms reduction and projection related to the Poisson structures are synonyms in this paper.

Definition 2.2. Let M be a manifold, $\eta \in \Gamma(\bigwedge^2 TM)$ be a bivector field (from now on we shall skip the last word). We consider η as a homomorphism:

$$\eta : T^*M \to TM,$$

obtained by the contraction on the first index and define the (generalized) distribution of characteristic subspaces $\chi^\eta \subset TM$ by

$$\chi^\eta_x = \operatorname{im} \eta_x, \quad x \in M.$$

Set $\operatorname{rank} \eta_x = \dim \operatorname{im} \eta_x$, $\operatorname{rank} \eta = \max_{x\in M} \operatorname{rank} \eta_x$ and $R^\eta = \{x \in M | \operatorname{rank} \eta_x = \operatorname{rank} \eta\}$. We say that η is nondegenerate if it is an isomorphism or, equivalently, $\chi^\eta = TM$.

Clearly, $R^\eta \subset M$ is an open dense set.

Definition 2.3. Let M be a manifold, $\mathcal{K}$ be a foliation on M such that the factor space $M' = M/\mathcal{K}$ is a manifold, and let $p : M \to M'$ be the canonical projection. We say that a bivector $\eta \in \Gamma(\bigwedge^2 TM)$ is projectable via p if there exists a bivector $\eta' \in \Gamma(\bigwedge^2 TM')$ (called the projection of η) such that

$$\eta'_{x'} = p_*\eta_x$$

for any $x' \in M'$, $x \in p^{-1}(x')$.

Definition 2.4. A bivector $\eta \in \Gamma(\bigwedge^2 TM)$ is called Poisson if the operation

$$\{f, g\}^\eta = \eta(f)g, \quad f, g \in \mathcal{E}(M),$$

where we put $\eta(f) = \eta(\mathrm{d}f)$, satisfies the Jacobi identity. The operation $\{,\}^\eta$ is called the Poisson bracket, the vector fields $\eta(f)$ are called hamiltonian.

Proposition 2.5 (e.g. [15]). *A bivector η is Poisson iff $[\eta, \eta] = 0$, where $[,]$ is the Schouten bracket on multivector fields.*

Theorem 2.6 ([12]). *If η is a Poisson bivector its generalized distribution of characteristic subspaces χ^η is completely integrable, i.e. there exists a generalized foliation $\mathcal{S}$ on M such that $T_x\mathcal{S} = \chi^\eta_x$ for any $x \in M$. The restriction $\eta|_S$ of η to any leaf S of $\mathcal{S}$ is a correctly defined nondegenerate Poisson bivector.*

Definition 2.7. The leaves of the generalized foliation $\mathcal{S}$ are called symplectic leaves of η.

The definition is motivated by the fact that the inverse to a nondegenerate Poisson bivector 2-form is symplectic.

Definition 2.8. A function $f \in \mathcal{E}(U)$ over an open set $U \subset M$ is called a Casimir function for η if $\eta(f) \equiv 0$. The set of all Casimir functions for η over U will be denoted by $Z^\eta(U)$.

Geometrically speaking the Casimir functions are those constant on the symplectic leaves of maximal dimension.

Definition 2.9. A set $Z \subset Z^\eta(U)$ of Casimir functions over $U \subset M$ is called complete if there exist $f_1, \ldots, f_k \in Z$, where $k = \operatorname{corank}\eta := \dim M - \operatorname{rank}\eta$, such that their differentials are independent on $U \cap R^\eta$.

In other words Z is complete iff the common level sets of functions from Z coincide with the symplectic foliation on $U \cap R^\eta$. It is clear that $Z^\eta(U)$ is complete for sufficiently small U.

Definition 2.10. A set $I \subset \mathcal{E}(U)$ of functions over $U \subset M$ is called complete involutive for η if: (1) $\{f, g\}^\eta = 0 \ \ \forall f, g \in I$; (2) there exist $f_1, \ldots, f_s \in I$, where $s = \dim M - (1/2)\operatorname{rank}\eta$, such that their differentials are independent on $U \cap R^\eta$.

If I is a complete involutive set over U, then the set $I \cap Z^\eta(U) \subset Z^\eta(U)$ is complete in the sense of Definition 2.9. Any such set I cuts a foliation of $U \cap R^\eta$ of dimension $(1/2)\operatorname{rank}\eta$ which is lagrangian in any symplectic leaf (of maximal dimension).

Definition 2.11. A map $\mu : (M, \eta) \to (M', \eta')$ between two Poisson manifolds is called Poisson if for any $f, g \in \mathcal{E}(M')$:

$$\mu^*\{f, g\}^{\eta'} = \{\mu^* f, \mu^* g\}^\eta,$$

or, equivalently, $\mu_*\eta_x = \eta'_{\mu(x)}$ for any $x \in M$.

Proposition 2.12 ([22], Lemma 1.2). *If $\mu : (M, \eta) \to (M_1, \eta_1)$ is a Poisson map, then the trajectory of any hamiltonian field $\eta_1(f)$ is the projection via μ of the trajectory of hamiltonian field $\eta(\mu^* f)$.*

Proposition 2.13. *Let $p : M \to M'$ be as in Definition 2.3 and assume that η is a Poisson bivector on M. Then the bivector η is projectable via p iff for any open set $U \subset M'$ the subspace $p^*\mathcal{E}(U) = \{p^* f | f \in \mathcal{E}(U)\} \subset \mathcal{E}(P^{-1}(U))$ is a Lie subalgebra with respect to $\{,\}^{\eta|U}$. If η is projectable and η' is the projection, then η' is a Poisson bivector and $p : (M, \eta) \to (M', \eta')$ is a Poisson map.*

Proof. Let $(U, \{\varphi_j\})$ be a coordinate map on M'. Since $p^*\mathcal{E}(U)$ is a subalgebra, $\{p^*\varphi_i, p^*\varphi_j\}^\eta = p^*c^{ij}$ for some function $c^{ij} \in \mathcal{E}(U)$. It is easily seen that c^{ij} transforms tensorially under coordinate changes, i.e. represents some bivector η' on M'. The remaining part of the proof is almost immediate consequence of the definitions. □

Here is another criterion of projectability.

Theorem 2.14 (Libermann–Weinstein criterion of projectability [14,22]). *Let $p : M \to M'$ and $\mathcal{K}$ be as in Definition 2.3 and let η be a nondegenerate Poisson bivector on M. Write $N\mathcal{K} \subset T^*M$ for the conormal bundle to the foliation $\mathcal{K}$. Then η is projectable via p iff the distribution $\eta(N\mathcal{K}) \subset TM$, which is the skew-orthogonal complement to the distribution $T\mathcal{K}$, is completely integrable.*

Corollary 2.15 ([22]). *Let $p : M \to M'$, $\mathcal{K}$, and η be as in the assumption of Theorem 2.14. Assume that η is projectable and that the foliation $\mathcal{K}''$ tangent to the distribution $\eta(N\mathcal{K})$ is such that the factor space $M'' = M/\mathcal{K}''$ is a manifold. Then η is also projectable to M'' via the canonical projection $p'' : M \to M''$.*

Proof. Follows from the fact that in the nondegenerate case $\eta(N[\eta(N\mathcal{K})]) = T\mathcal{K}$, i.e. the distributions $T\mathcal{K}$ and $\eta(N\mathcal{K})$ are the skew-orthogonal complements of each other. □

Definition 2.16 ([22]). Let η be a nondegenerate Poisson bivector on M and let $\mathcal{K}'$, $\mathcal{K}''$ be foliations on M such that $\eta(N\mathcal{K}') = T\mathcal{K}''$ and the factor spaces $M' = M/\mathcal{K}'$, $M'' = M/\mathcal{K}''$ are manifolds. The pair (η', η''), where $\eta' = p'_*\eta$, $\eta'' = p''_*\eta$ are the projections of η via the canonical projections $p' : M \to M'$ and $p'' : M \to M''$ respectively, is called a dual pair of Poisson bivectors.

The situation can be expressed by the following diagram:

$$\begin{tikzcd}[column sep=small] & (M,\eta) \arrow[dl, "{(p',p'_*)}"'] \arrow[dr, "{(p'',p''_*)}"] & \\ (M' = M/\mathcal{K}', \eta' = p'_*\eta) & & (M'' = M/\mathcal{K}'', \eta'' = p''_*\eta), \end{tikzcd}$$

where $\eta(N\mathcal{K}') = T\mathcal{K}''$.

Example 2.17. Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$. Assume it is acting on a Poisson manifold (M, η), in particular a Lie algebra homomorphism $\rho : \mathfrak{g} \to \Gamma TM$ is given (the space of vector fields is endowed with the commutator Lie bracket).

The action is called hamiltonian if there exists a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathcal{E}(M)$ such that the following diagram is commutative:

$$\begin{array}{lcl} \mathfrak{g} & \xrightarrow{\psi} & \mathcal{E}(M) \\ \| & & \downarrow \eta(\cdot) \\ \mathfrak{g} & \xrightarrow{\rho} & \Gamma TM. \end{array}$$

where $\eta(\cdot)$ is the Lie algebra homomorphism of taking the hamiltonian vector field (see Definition 2.4). The map $\mu : x \mapsto \varphi_x : M \to \mathfrak{g}^*$ defined by $\varphi_x(v) = \psi(v)(x)$, $v \in \mathfrak{g}$, is called the moment map.

Assume that G acts on M by Poisson maps and that M/G is a manifold. Then by Proposition 2.13 the bivector η is projectable via the canonical projection $M \to M/G$. If moreover, η is nondegenerate and the action is hamiltonian its orbits are skew-orthogonal to the fibers of the moment map. This last is a Poisson map from (M, η) to $(\mathfrak{g}^*, \eta_{\mathrm{can}})$, where η_{can} is the canonical linear Poisson bivector on $\mathfrak{g}^*$. In case of a locally free action μ is a submersion (see the proof of Corollary 2.20) and we get a dual pair $(\eta', \eta_{\mathrm{can}})$.

We complete the section by a series of propositions that will be crucial for the subsequent part of the paper. Proposition 2.18 and Corollaries 2.19 and 2.20 are classical.

Proposition 2.18. *We retain the notations of* Definition 2.16. *Assume that* (η', η'') *is a dual pair of Poisson bivectors. Then*

(a) *The distribution* $D_x = T_x\mathcal{K}' + T_x\mathcal{K}'' \subset T_xM, x \in M$, *is of constant dimension on an open dense set* $R \subset M$ *and is completely integrable on* R.
(b) *The foliation* $\mathcal{D}$ *tangent to* D *on* R *is the pull-back of the foliations of symplectic leaves* $\mathcal{S}', \mathcal{S}''$ *of maximal dimension of the bivectors* η', η'', *respectively*:

$$(p')^*\mathcal{S}' = \mathcal{D} = (p'')^*\mathcal{S}''.$$

(c) corank η' = corank η''.

Proof. The constancy of dimension on an open dense set follows from analyticity of all objects. Item (b) is a consequence of Proposition 2.12 and of skew-orthogonality of the foliations $\mathcal{K}', \mathcal{K}''$. Also, (b) implies integrability of D and (c). □

Corollary 2.19. *We retain the assumptions of* Proposition 2.18. *Let* $U' \subset M'$, $U'' \subset M''$ *be open sets such that the sets of Casimir functions* $Z' := Z^{\eta'}(U')$, $Z'' := Z^{\eta''}(U'')$ *are complete* (*see* Definition 2.9) *and* $U := (p')^{-1}(U') \cap (p'')^{-1}(U'') \neq \emptyset$. *Put* $((p')^*Z')|_U = \{((p')^*f)|_U | f \in Z'\}$ *and* $((p'')^*Z'')|_U = \{((p'')^*g)|_U | g \in Z''\}$. *Then*

$$((p')^*Z')|_U = ((p'')^*Z'')|_U. \tag{2.1}$$

Corollary 2.20. *Assume that a Lie group* G *is acting in the hamiltonian way on a nondegenerate Poisson manifold* (M, η) (*see* Example 2.17). *Assume moreover, that this action is locally free* (*the stabilizer of any point is at most discrete*) *and that* M/G *is a manifold. Then for any* $x' \in M/G$ *we have* corank $\eta'_{x'}$ = rank G, *where* η' *is the projection of* η *via the canonical map* $M \to M/G$.

Proof. It is well-known that the image of the differential at a point $x \in M$ of the moment map $\mu : M \to \mathfrak{g}^*$ coincides with the annihilator in $\mathfrak{g}^*$ of $\mathfrak{g}^x \subset \mathfrak{g}$, where $\mathfrak{g}^x$ is the Lie algebra of the stabilizer of x (see [11], Lemma 2.1). Thus in our situation when the stabilizer is discrete μ is a submersion. By Proposition 2.18 corank of η' coincides with the one of η_{can}, i.e. with rank of the Lie group G. □

Here is a generalization of this result to the case of degenerate Poisson bivector η.

Proposition 2.21. *Let* η *be a regular Poisson bivector* (*i.e.* rank η_x = const) *on* M *and let a Lie group* G *act locally freely on* M *in such a way that* M/G *is a manifold. Given* *a symplectic leaf* $S \subset M$, *write* $G_S \subset G$ *for its stabilizer, i.e. for a subgroup defined by* $G_S S \subset S$. *Fix* S *and assume that*:

(1) G *acts by Poisson maps, i.e. the action preserves* η;
(2) *the action induces a transitive action on the space of symplectic leaves*;
(3) *the induced action of* G_S *on* $(S, \eta|_S)$ *is hamiltonian.*

Then

(a) *if* $\hat{S}$ *is any symplectic leaf, the stabilizers* $G_S, G_{\hat{S}}$ *are conjugate*;
(b) *the induced action of* $G_{\hat{S}}$ *on* $\hat{S}$ *is hamiltonian*;
(c) η *is projectable via the canonical map* $M \to M/G$ *and* $\operatorname{corank} \eta'_{x'} = \operatorname{rank} G_S$ *for any* $x' \in M/G$, *where* η' *is the projection.*

Proof. Since any two points on any symplectic leaf S of η can be connected by a finite number of hamiltonian trajectories and since the action preserves η, it follows from Proposition 2.12 that the image gS, $g \in G$, is again a symplectic leaf. Now, assumption (2) implies that for any $S, \hat{S}$ there exists $a \in G$ such that $aS = \hat{S}$, hence $G_S = \{g \in G | gS = S\} = \{g \in G | ga^{-1}\hat{S} = a^{-1}\hat{S}\} = \{g \in G | aga^{-1}\hat{S} = \hat{S}\} = a^{-1}G_{\hat{S}}a$.

To prove (b) let us consider the induced action $\rho_{\hat{S}} : \mathfrak{g}_{\hat{S}} \to \Gamma T\hat{S}$ of the Lie algebra of the stabilizer $G_{\hat{S}}$ on $\hat{S}$. Its hamiltonicity follows from the following commutative diagram:

$$
\begin{array}{llll}
\mathfrak{g}_{\hat{S}} & \xrightarrow{\rho_{\hat{S}}} & \Gamma T\hat{S} & = \Gamma T\hat{S} \\
\downarrow \operatorname{Ad} a & & \uparrow L_{a*} & \| \\
\mathfrak{g}_S & \xrightarrow{\rho_S} & \Gamma T S & \Gamma T\hat{S} \\
\| & & \uparrow \eta(\cdot) & \uparrow \eta(\cdot) \\
\mathfrak{g}_S & \xrightarrow{\psi_S} & \mathcal{E}(S) & \mathcal{E}(\hat{S}) \\
\| & & \uparrow L_a^* & \| \\
\mathfrak{g}_S & \xrightarrow{L^*_{a^{-1}} \circ \psi_S} & \mathcal{E}(\hat{S}) & = \mathcal{E}(\hat{S}),
\end{array}
$$

where all the maps are Lie algebra homomorphisms, ψ_S is one existing by assumption (3), L_a denotes the left multiplication by a.

Projectability of η follows from (1) and from Proposition 2.13. Condition (2) guarantees that the projection η' of η via the map $M \to M/G$ coincides with the projection $(\eta|_S)'$ of the restricted Poisson bivector $\eta|_S$ via the map $S \to S/G_S = M/G$. Taking into account assumption (3) we can apply Corollary 2.20 to the action of G_S on $(S, \eta|_S)$. This proves (c). □

Proposition 2.22. *We retain the notations of* Definition 2.16. *Let* (η', η'') *be a dual pair of Poisson bivectors, let* $U' \subset M'$, $U'' \subset M''$ *be open sets such that* $U := (p')^{-1}(U') \cap (p'')^{-1}(U'') \neq \emptyset$ *and let* $I' \subset \mathcal{E}(U')$, $I'' \subset \mathcal{E}(U'')$ *be complete involutive sets of functions for* η', η'' *respectively. Put* $((p')^*I')|_U = \{((p')^*f)|_U | f \in I'\}$ *and* $((p'')^*I'')|_U = \{((p'')^*g)|_U | g \in I''\}$. *Then the space* $I := ((p')^*I')|_U + ((p'')^*I'')|_U$ *is a complete involutive set of functions for* η.

Proof. We first notice that since $\mathcal{K}'$ and $\mathcal{K}''$ are skew-orthogonal, $\{(p')^*f, (p'')^*g\}^\eta = 0$ for any $f \in \mathcal{E}(U')$, $g \in \mathcal{E}(U'')$. Together with the Poisson property for p' and p'' this shows that I is an involutive set of functions with respect to η. Now we only need to calculate its "functional dimension".

Let us choose a "functional basis" $\{f_1, \dots, f_{s'}\}$ of I' such that $f_1, \dots, f_{r'} \in Z^{\eta'}(U')$ and any "functional basis" $\{g_1, \dots, g_{s''}\}$ of I''. Then the functions $(p')^* f_{r'+1}, \dots, (p')^{\,ast} f_{s'}$, $(p'')^* g_1, \dots, (p'')^* g_{s''}$ are functionally independent on an open dense subset of U since

$$\{(p')^* f|_U | f \in \mathcal{E}(U')\} \cap \{(p'')^* g|_U | g \in \mathcal{E}(U'')\} = Z,$$

where Z denotes the set (2.1). Now, one has

$$s' - r' = \tfrac{1}{2}\mathrm{rank}\,\eta' = \tfrac{1}{2}(\dim \mathcal{K}'' - \dim \mathcal{K}'' \cap \mathcal{K}'),$$
$$s'' = \tfrac{1}{2}\mathrm{rank}\,\eta'' + \mathrm{corank}\,\eta'' = \tfrac{1}{2}(\dim \mathcal{K}' - \dim \mathcal{K}'' \cap \mathcal{K}') + \dim \mathcal{K}'' \cap \mathcal{K}'$$

and, finally

$$s' - r' + s'' = \tfrac{1}{2}(\dim \mathcal{K}'' + \dim \mathcal{K}') = \tfrac{1}{2}\dim M. \qquad \square$$

3. Preliminaries on bi-Poisson structures

Definition 3.1. A pair (η_1, η_2) of linearly independent bivectors on a manifold M is called Poisson if $\eta^t := t_1\eta_1 + t_2\eta_2$ is a Poisson bivector for any $t = (t_1, t_2) \in \mathbb{K}^2$; the whole family of Poisson bivectors $\{\eta^t\}_{t\in\mathbb{K}^2}$ is called a bi-Poisson structure. We define the trivial bi-Poisson structure as a family consisting of the zero bivector.

A bi-Poisson structure $\{\eta^t\}$ (we shall often skip the parameter space in the notations) can be viewed as a two-dimensional vector space of Poisson bivectors, the Poisson pair (η_1, η_2) as a basis in this space. Of course, the basis can be changed.

Definition 3.2. A bi-Poisson structure $\{\eta^t\}$ is called Jordan at a point $x \in M$ if $\mathrm{rank}\,\eta^t_x = \dim M$ for some t. A bi-Poisson structure is called micro-Jordan if it is Jordan at any point of some open dense subset in M.

The terminology is due to Gelfand and Zakharevich [8,23] who reduced the analysis of a bi-Poisson structure at a point to the study of a pencil of operators and applied the classical classificational results. These last say that any pencil is built of the irreducible ones, the so-called Jordan and Kronecker blocks. The above definition corresponds to the case when only the Jordan blocks are present.

The theory of pencils of operators is well understood over the field of complex numbers. We shall also need some notions related to the complexification matters.

Notation 3.3. If M is a real manifold (recall that all objects are real-analytic) we denote by $\tilde{M}$ some complexification of M, i.e. a complex manifold $\tilde{M}$ such that M is embedded in $\tilde{M}$ as a totally real submanifold. The complex structure near M is defined uniquely up to a biholomorphic map preserving M (see [3]), thats why we use the same notation $\tilde{M}$ for possibly different complexifications. Given any tensor η on M, we write $\tilde{\eta}$ for its complexification, which is a holomorphic tensor defined on $\tilde{M}$ (the last should be shrinked if needed).

For any real bi-Poisson structure $\{\eta^t = t_1\eta_1 + t_2\eta_2\}$ on M we denote by $\tilde{\eta}^t$ its complexification, i.e. the holomorphic bi-Poisson structure $\{\tilde{\eta}^t = t_1\tilde{\eta}_1 + t_2\tilde{\eta}_2 | t = (t_1, t_2) \in \mathbb{C}^2\}$ on $\tilde{M}$.

If M and $\{\eta^t\}$ are a priori holomorphic we put $\tilde{M} = M$, $\{\tilde{\eta}^t\} = \{\eta^t\}$, etc., i.e. tilde for holomorphic objects will denote themselves (not the complexification of the underlying real-analytic objects).

Definition 3.4. Let $\{\eta^t\}$ be a micro-Jordan bi-Poisson structure on M. Put $E(x) = \{t \in \mathbb{C}^2 | \operatorname{rank}_{\mathbb{C}} \tilde{\eta}^t_x < \dim_{\mathbb{C}} \tilde{M}\} \subset \mathbb{C}^2, x \in M$. This set is called exceptional for $\{\eta^t\}$ at x. If $E = E(x)$ does not depend on x the structure $\{\eta^t\}$ is called dull.

This terminology is due to Zakharevich and is motivated by the fact that the constancy of $E(x)$ implies the constancy of the eigenvalues for the recursion operator $\eta_1^{-1} \circ \eta_2^{-1}$, (assume that the basis η_1, η_2 is so chosen that η_2 is nondegenerate), i.e. the situation is far from being of interest in the theory of integrable systems in which these eigenvalues appear as the first integrals.

It is clear that $E(x)$ consists of a finite number of one-dimensional subspaces in $\mathbb{C}^2$.

Definition 3.5. Let $\{\eta^t\}$ be a bi-Poisson structure on M. It is called Kronecker at a point $x \in M$ if $\operatorname{rank}_{\mathbb{C}} \tilde{\eta}^t_x$ is constant with respect to $t \in \mathbb{C}^2 \setminus \{0\}$. We say that $\{\eta^t\}$ is micro-Kronecker if it is Kronecker at any point of some open dense set in M. In particular the trivial bi-Poisson structure is micro-Kronecker.

Again this terminology is due to Zakharevich and is motivated by the fact that under the above rank assumptions the corresponding pencil of operators (see the discussion after Definition 3.2) contains only the Kronecker blocks.

Definition 3.6. Let $p : M \to M'$ be as in Definition 2.3 and let $\{\eta^t = t_1\eta_1 + t_2\eta_2\}$ be a bi-Poisson structure on M. We say that it is projectable via p if so is the bivector η^t for any t. The family $\{(\eta^t)' = t_1\eta'_1 + t_2\eta'_2\}$ consisting of the projections of η^t, which is a bi-Poisson structure on M' under the condition that the bivectors η'_1, η'_2 are linearly independent or trivial (see Proposition 2.13), is called the projection of $\{\eta^t\}$.

Now we are able to formulate the main question of this paper: when the projection of a (projectable) dull micro-Jordan bi-Poisson structure is micro-Kronecker? We shall answer it in the next section for some particular cases of locally free bi-Poisson actions. Now we want to present a result which shows why the micro-Kronecker structures are interesting and which will be effectively used later.

Proposition 3.7. *Let $\{\eta^t\}$ be a micro-Kronecker bi-Poisson structure on M. Assume that an open set $U \subset M$ is such that the set $Z^{\eta^t}(U)$ of Casimir functions for η^t over U is complete* (*see* Definition 2.9) *for any $t \neq 0$. Then the set*

$$Z^{\{\eta^t\}}(U) := \sum_{t\neq 0} Z^{\eta^t}(U),$$

is a complete involutive set of functions for any $\eta^t \neq 0$ (*see* Definition 2.10). (*Here and subsequently in similar situations we understand the sum as the algebraic sum of linear* (*sub*)*spaces of functions in the linear space of all functions. In other words this sum coincides with the linear span $\langle Z^{\eta^t}(U)|t \neq 0\rangle$. Of course, it is enough to sum over a sufficiently large finite set of indices t.*)

We shall call the functions from $Z^{\{\eta^t\}}$ the first integrals of the bi-Poisson structure $\{\eta^t\}$. The reader is referred to a celebrated paper of Bolsinov [2] for the proof of completeness. Although the involutivity of this set was known and extensively used since the end of the 80's the author was not able to find its proof and gave a version of it in [16].

Example 3.8 (Method of the argument translation). Let $\mathfrak{g}$ be a Lie algebra with codim Sing $\mathfrak{g}^* \geq 3$, where Sing $\mathfrak{g}^* \subset \mathfrak{g}^*$ is the algebraic set of all coadjoint orbits of nonmaximal dimension (in particular $\mathfrak{g}$ can be any semisimple). Let $\eta_1 = \eta_{\text{can}}$ be a canonical linear Poisson bivector on $\mathfrak{g}^*$, and let $\eta_2 = \eta_{\text{can}}(a)$ be the Poisson bivector obtained by "freezing" η_{can} at a regular (i.e. belonging to $\mathfrak{g}^* \setminus \text{Sing}\,\mathfrak{g}^* = R^{\eta_{\text{can}}}$) element a. It is well-known that (η_1, η_2) is a Poisson pair and that the corresponding bi-Poisson structure $\{\eta^t_{\text{AT}}\}$ is micro-Kronecker (see [2,16,23]). The set of first integrals $Z^{\{\eta^t_{\text{AT}}\}}$ is functionally generated by $f_1(x+\lambda a), \dots, f_k(x+\lambda a)$, $\lambda \in \mathbb{K}$, where $f_1, \dots, f_k$ are the invariants of the coadjoint action.

4. A locally free bi-Poisson action of a Lie group on a dull micro-Jordan structure

Assumptions and notations 4.1. Let G be a real Lie group. We shall assume that it possesses the complexification, i.e. a complex Lie group $\tilde{G} = G^{\mathbb{C}}$ containing G as a real subgroup such that its Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{g}$ of G. In particular, G may be linear semisimple or compact.

Given a real dull micro-Jordan bi-Poisson structure $\{\eta^t = t_1\eta_1 + t_2\eta_2\}$ on a manifold M, we denote by $\tilde{M}$ a complexification of M such that the bivectors η_1, η_2 are extended to holomorphic Poisson bivectors $\tilde{\eta}_1, \tilde{\eta}_2$ (automatically forming a Poisson pair on $\tilde{M}$). We write $\{\tilde{\eta}^t\}$ for the holomorphic bi-Poisson structure $\{t_1\tilde{\eta}_1 + t_2\tilde{\eta}_2\}$, $e_1, \dots, e_N$ for the vectors in $\mathbb{C}^2$ spanning the lines of the exceptional set $E = \langle e_1\rangle \cup \dots \cup \langle e_N\rangle$ (see Definition 3.4), and $\tilde{\eta}^{e_1}, \dots, \tilde{\eta}^{e_N}$ for the corresponding exceptional bivectors.

We retain the convention that $\widetilde{(\cdot)} = (\cdot)$ for a holomorphic object $(\cdot)$ (cf. Notation 3.3).

The central result of this paper is the following.

Theorem 4.2. *We retain the above assumptions and notations. Assume a Lie group G is acting locally freely on a manifold M with a dull micro-Jordan bi-Poisson structure $\{\eta^t\}$, that* *this action is extended to a locally free action of* $\tilde{G}$ *on* $\tilde{M}$ (*in the complex case this extended action is the initial one*) *and that* M/G, $\tilde{M}/\tilde{G}$ *are manifolds. For any* $j = 1, \ldots, N$ *fix a symplectic leaf* S_j *of maximal dimension of the exceptional bivector* $\tilde{\eta}^{e_j}$ *and let* $\tilde{G}_j$ *denote its stabilizer*.

We make the following additional assumptions on the $\tilde{G}$*-action on* $\tilde{M}$:

(1) *it is bi-Poisson, i.e. preserves* $\tilde{\eta}_1, \tilde{\eta}_2$;
(2) *it induces a transitive action on the space of symplectic leaves of maximal dimension of any exceptional bivector* $\tilde{\eta}^{e_j}$;
(3) *the induced action of* $\tilde{G}_j$ *on* $(S_j, \tilde{\eta}^{e_j}|_{S_j})$ *is hamiltonian*;
(4) *the action of* $\tilde{G}$ *on* $(\tilde{M}, \tilde{\eta}^t)$, $t \in \mathbb{C}^2 \setminus E$, *is also hamiltonian.*

Then

- $\{\eta^t\}$ *is projectable via the canonical map* $p : M \to M/G$;
- *the projection* $\{(\eta^t)'\}$ *is a bi-Poisson structure under the condition that the bivectors* η_1', η_2' *are linearly independent or trivial*;
- $\{(\eta^t)'\}$ *is Kronecker at any point* $x' \in p(R^{\tilde{\eta}^{e_1}} \cup \cdots \cup R^{\tilde{\eta}^{e_N}}) \subset M/G$ *iff*

$$\operatorname{rank} \tilde{G} = \operatorname{rank} \tilde{G}_1 = \cdots = \operatorname{rank} \tilde{G}_N$$

(*recall that* R^η *stands for the regularity set of a bivector* η, *see* Definition 2.2).

Proof. It is clear that each η^t is projectable (since G acts by the Poisson maps with respect to η_1, η_2, see Proposition 2.13), and that $\{(\eta^t)'\}$ is a bi-Poisson structure provided η_1', η_2' are linearly independent or trivial.

By definition $\{(\eta^t)'\}$ is Kronecker at x' iff $\operatorname{corank}(\tilde{\eta}^t)'_{x'}$ is constant with respect to $t \neq 0$. Now it remains to use Corollary 2.20 to deduce that $\operatorname{corank}(\tilde{\eta}^t)'_{x'} = \operatorname{rank} \tilde{G}$ for $t \in \mathbb{C}^2 \setminus E$ and Proposition 2.21 to get $\operatorname{corank}(\tilde{\eta}^{e_j})'_{x'} = \operatorname{rank} \tilde{G}_j$, $j = 1, \ldots, N$. □

Corollary 4.3. *In the situation of the above theorem let* $\mu_t : M \to \mathfrak{g}^*$, $t \in \mathbb{K}^2 \setminus E$, *denote the moment map corresponding to* η^t. *Assume that* $\{(\eta^t)'\}$ *is Kronecker. Then*

(a) *the pull-back of the set of first integrals* $p^*\mathcal{F} := p^*(Z^{\{(\eta^t)'\}})$ (*see* Proposition 3.7) *is equal to*

$$p^*\mathcal{F} = \sum_{s \in \mathbb{K}^2 \setminus E} \mu_s^*(Z^{\eta_{\mathrm{can}}}),$$

where η_{can} *is the canonical linear Poisson bivector on the dual space* $\mathfrak{g}^*$ *to the Lie algebra of* G;

(b) *provided that* G *satisfies the condition* $\operatorname{codim} \operatorname{Sing} \mathfrak{g}^* \geq 3$ *of the argument translation method* (*see* Example 3.8), *one gets the following complete involutive with respect to any fixed* η^{t_0}, $t_0 \notin E$, *set of functions on* M:

$$\mathcal{G}^{t_0} := \sum_{s \in \mathbb{K}^2 \setminus E} \mu_s^*(Z^{\eta_{\mathrm{can}}}) + \mu_{t_0}^*(Z^{\{\eta^t_{\mathrm{AT}}\}}).$$

Proof. The Proof of (a) follows from Corollary 2.19 and from the definition of the first integrals; the proof of (b) is a consequence of (a) and Proposition 2.22. □

Example 4.4. Let $M = \mathbb{R}^{2N}$ with coordinates $\{p_j, q_j\}_{j=1}^N$, $\eta_1 = (\partial/\partial p_1) \wedge (\partial/\partial q_1) + \cdots + (\partial/\partial p_N) \wedge (\partial/\partial q_N)$, $\eta_2 = a_1(\partial/\partial p_1) \wedge (\partial/\partial q_1) + \cdots + a_N(\partial/\partial p_N) \wedge (\partial/\partial q_N)$, where $a_1, \dots, a_N$ are different real numbers. Then the family $\{\eta^t\} := \{t_1\eta_1 + t_2\eta_2\}$, $t = (t_1, t_2) \in \mathbb{R}^2$, is a dull micro-Jordan bi-Poisson structure with the exceptional set $E = \langle(a_1, -1)\rangle \cup \cdots \cup \langle(a_N, -1)\rangle \subset \mathbb{C}^2$, the exceptional bivectors $\tilde{\eta}^{e_j} = \tilde{\eta}^{(a_j,-1)}$ and the corresponding foliations of symplectic leaves $\mathcal{S}_j = \{P_j = \text{const}, Q_j = \text{const}\}$, $j = 1, \dots, N$, where $\{P_j, Q_j\}_{j=1}^N$, $P_j = p_j + \mathrm{i}\hat{p}_j$, $Q_j = q_j + \mathrm{i}\hat{q}_j$, are the holomorphic coordinates on $M^{\mathbb{C}} = \mathbb{C}^{2N}$.

Assume $G = \mathrm{SL}(2, \mathbb{R})$ is acting on $\mathbb{R}^2$ in a standard linear way and that this action is extended to $M = \mathbb{R}^{2N}$ diagonally. It is easy to see that all these data satisfy the assumptions of Theorem 4.2. Moreover, the stabilizers $\tilde{G}_1, \dots, \tilde{G}_N \subset \tilde{G} = \mathrm{SL}(2, \mathbb{C})$ of fixed symplectic leaves $S_j = \{P_j = b_j, Q_j = c_j\} \subset \mathcal{S}_j$, $j = 1, \dots, N$, which coincide with the stabilizers of the vectors $\left[\begin{smallmatrix} b_j \\ c_j \end{smallmatrix}\right]$ under the standard linear $\tilde{G}$-action, are one-dimensional, consequently abelian and have rank 1 equal to rank of $\tilde{G}$. Hence the reduced bi-Poisson structure $\{(\eta^t)'\}$ is Kronecker on the regular part of the variety M/G.

The calculations show that the moment map which corresponds to η^t is

$$\mu_t : (p, q) \mapsto \begin{bmatrix} z_1 = -\sum_j \dfrac{p_j q_j}{t_1 + a_j t_2} \\[2ex] z_2 = -(1/2)\sum_j \dfrac{q_j^2}{t_1 + a_j t_2} \\[2ex] z_3 = (1/2)\sum_j \dfrac{p_j^2}{t_1 + a_j t_2} \end{bmatrix} : \mathbb{R}^{2N} \to (\mathrm{sl}(2, \mathbb{R}))^*,$$

and that the Casimir function of η_{can} on $(\mathrm{sl}(2, \mathbb{R}))^*$ is $f = z_1^2 + 4z_2z_3$. Introducing the affine parameter $r = -(t_1/t_2)$ we get an involutive family of functions on M:

$$p^*\mathcal{F} = \sum_{r\in\mathbb{R}} \left\langle \left(\sum_{j=1}^N \frac{p_j q_j}{r - a_j}\right)^2 - \left(\sum_{j=1}^N \frac{q_j^2}{r - a_j}\right)\left(\sum_{j=1}^N \frac{p_j^2}{r - a_j}\right)\right\rangle$$

(here $p : M \to M/G$ is the canonical map). Expanding this expression with respect to the powers of $r - a_j$ and calculating the coefficients corresponding to the first powers we obtain the following functions generating $p^*\mathcal{F}$:

$$\sum_{k=1, k\neq j}^N \frac{(p_k q_j - p_j q_k)^2}{a_k - a_j}, \quad j = 1, \dots, N.$$

There is one relation between these functions. By Corollary 4.3(b) applied with the choice $t_0 = (1, 0)$ (i.e. $\eta^{t_0} = \eta_1$) $p^*\mathcal{F}$ can be completed by the function $\mu^*_{(1,0)}g$, where $g =$

$z_1z_1^0 + 2z_2z_3^0 + 2z_3z_2^0$ is obtained from f by the shift in the direction of an element $z^0 = (z_1^0, z_2^0, z_3^0) \in (\mathrm{sl}(2,\mathbb{R}))^*$:

$$f(z + \lambda z^0) = f(z) + 2\lambda g(z) + \lambda^2 f(z^0), \quad \lambda \in \mathbb{R}.$$

Finally, we get the following complete involutive (with respect to a standard Poisson bracket) set of functions on $\mathbb{R}^{2N}$:

$$\sum_{k=1,k\neq j}^{N} \frac{(p_kq_j - p_jq_k)^2}{a_k - a_j}, \quad j = 1, \ldots, N-1,$$
$$z_1^0 \sum_{j=1}^{N} p_jq_j + z_2^0 \sum_{j=1}^{N} p_j^2 + z_3^0 \sum_{j=1}^{N} q_j^2,$$

where z_i^0, $i = 1, 2, 3$, are any constants simultaneously not equal to 0.

5. Main example: diagonal action of a Lie group on the product of *N* copies of the dual space to its Lie algebra

Let G be a complex Lie group, $\mathfrak{g}$ its Lie algebra. There is a natural coadjoint action of the direct product $G^{\times N}$ of N copies of G on $(\mathfrak{g}^*)^{\times N}$ which restricts to $G \subset G^{\times N}$ embedded diagonally. Let $p_j : (\mathfrak{g}^*)^{\times N} \to \mathfrak{g}^*$, $j = 1, \ldots, N$, denote the natural projection to the jth component and let η be the canonical linear Poisson bivector (c.l.P.b.) on $\mathfrak{g}^*$. Then the c.l.P.b. $\eta^{\times N}$ on $(\mathfrak{g}^*)^{\times N}$ has the decomposition $\eta^{\times N} = \eta_{(1)} + \cdots + \eta_{(N)}$, where $\eta_{(j)}$, $j = 1, \ldots, N$, is the unique Poisson bivector on $(\mathfrak{g}^*)^{\times N}$ defined by the condition $p_{j*}\eta_{(j)} = \eta$, $p_{i*}\eta_{(j)} = 0$, $i \neq j$.

Proposition 5.1. *Fix a coadjoint orbit* $\mathcal{O} = G^{\times N}(x_1, \ldots, x_N) \subset (\mathfrak{g}^*)^{\times N}$ *of an element* $(x_1, \ldots, x_N) \in (\mathfrak{g}^*)^{\times N}$ *and different numbers* $a_1, \ldots, a_N \in \mathbb{C}$. *Then*

(a) *the bivectors* $\eta^{\times N}$ *and* $\eta^{a\times N} := a_1\eta_{(1)} + \cdots + a_N\eta_{(N)}$ *form a Poisson pair on* $(\mathfrak{g}^*)^{\times N}$;
(b) *they are G-invariant*;
(c) *they have the natural restrictions* (*being Poisson bivectors*) $\eta_1 = \eta^{\times N}|_{\mathcal{O}}$, $\eta_2 = \eta^{a\times N}|_{\mathcal{O}}$ *to* $\mathcal{O}$;
(d) *the family* $\{\eta^t = t_1\eta_1 + t_2\eta_2\}$ *is a dull micro-Jordan bi-Poisson structure on* $\mathcal{O}$ *with the exceptional set* $E = \langle(a_1, -1)\rangle \cup \cdots \cup \langle(a_N, -1)\rangle$;
(e) *for any* $j = 1, \ldots, N$ *the symplectic foliation* $\mathcal{S}_j$ *of the exceptional bivector* $\eta^{e_j} = \eta^{(a_j,-1)}$ *coincides with the foliation of fibers of the natural projection* $p_j|_{\mathcal{O}} : \mathcal{O} = Gx_1 \times \cdots \times Gx_N \to Gx_j$.

Proof. Item (a) follows from Proposition 2.5 since $[\eta_{(i)}, \eta_{(j)}] = 0$ for any $i, j = 1, \ldots, N$. The first bivector is G-invariant by definition. The invariance of the second one follows from the G-equivariance of the projections p_j and from the invariance of η.

The restriction of $\eta^{\times N}$ to $\mathcal{O}$ is simply the restriction to a symplectic leaf. Moreover, any $\eta_{(j)}$ is tangent to the leaves of any projection p_i, $i \neq j$, and to $p_j^{-1}(Gx_j)$, i.e. $\eta_{(j)}$ also has the restriction to $\mathcal{O}$. This implies (c).

Since η_1 is nondegenerate (as any restriction of a Poisson bivector to a symplectic leaf), $\{\eta^t\}$ is micro-Jordan. Obviously, the only degenerate bivectors in this family are those proportional to η^{e_j}, $j = 1, \dots, N$, and the corresponding characteristic distributions satisfy the equalities $\chi^{\eta^{e_j}} = \sum_{i \neq j}(\chi^{\eta_{(i)}})|_{\mathcal{O}}$, which complete the proof. □

The main result of this section (Theorem 5.3) will study the reduction of the bi-Poisson structure $\{\eta^t\}$ on $G^{\times N}$-orbits under the action of G. Now we shall specify the class of orbits under consideration.

Definition 5.2. An orbit $\mathcal{O} = G^{\times N}(x_1, \dots, x_N) \subset (\mathfrak{g}^*)^{\times N}$ is called admissible if:

(1) There exist elements $x_1' \in Gx_1, \dots, x_N' \in Gx_N$ such that their stabilizers $G_{\mathfrak{g}^*}^{x_j'} \subset G$, $j = 1, \dots, N$, have discrete intersection; equivalently:

$$\mathfrak{g}_{\mathfrak{g}^*}^{x_1'} \cap \cdots \cap \mathfrak{g}_{\mathfrak{g}^*}^{x_N'} = \{0\}.$$

(2) The stabilizers $G_j := G_{\mathfrak{g}^*}^{x_j} \subset G$, $j = 1, \dots, N$, have all the same rank equal to the rank of G:

$$\operatorname{rank} G_1 = \cdots = \operatorname{rank} G_N = \operatorname{rank} G.$$

We postpone the discussion of the question which orbits are admissible to the end of this section (see Theorem 5.7 and Remark 5.9); here we mention only that the admissibility holds for generic orbits in the semisimple case.

Now we formulate the second main result of this paper.

Theorem 5.3. *Let $\mathcal{O} \subset (\mathfrak{g}^*)^{\times N}$ be an admissible $G^{\times N}$-orbit and let $M \subset \mathcal{O}$ be an open set such that M/G is a manifold. Then the bi-Poisson structure $\{\eta^t\}|_M$ is projectable via the canonical map $p : M \to M/G$ and the projection $\{(\eta^t)'\}$ is a micro-Kronecker bi-Poisson structure* (*see* Definition 3.5) *on $M' = M/G$. More precisely, $\{(\eta^t)'\}$ is Kronecker at any $x' \in M' \setminus p(\mathcal{N})$, where $\mathcal{N} \subset (\mathfrak{g}^*)^{\times N}$ is the algebraic set of all elements with a nondiscrete G-stabilizer.*

Proof. Of course, this proof will use Theorem 4.2. Now we shall check that the G-action on $\{\eta^t\}$ satisfies the assumptions of this theorem.

First, we note that since the G-stabilizer $G_{(\mathfrak{g}^*)^{\times N}}^{(x_1,\dots,x_N)}$ of a point $(x_1, \dots, x_N) \in (\mathfrak{g}^*)^{\times N}$ is equal to the intersection $G_{\mathfrak{g}^*}^{x_1} \cap \cdots \cap G_{\mathfrak{g}^*}^{x_N}$, condition (1) in definition of admissibility guarantees that the G-action is locally free.

The G-invariance of $\{\eta^t\}$ was proved in Proposition 5.1(a), so we get assumption (1) of Theorem 4.2. To check assumption (2) recall (see Proposition 5.1(e)) that the symplectic foliation of the exceptional bivector η^{e_j} coincides with $\{Gx_1 \times \cdots \times Gx_{j-1} \times x \times$

$Gx_{j+1} \times \cdots \times Gx_N | x \in Gx_j\}$. Since G is acting transitively on Gx_j, the same is true for the induced G-action on the leaves of this foliation.

Now, let us prove the hamiltonicity of the G-action on M with respect to η^t, $t \in \mathbb{C}^2 \setminus E$. The commutativity of the following diagram is standard:

$$\begin{array}{ccc} \mathfrak{g}^{\times N} & \xrightarrow{i} & \mathcal{E}((\mathfrak{g}^*)^{\times N}) \\ \| & & \downarrow \eta^{\times N}(\cdot) \\ \mathfrak{g}^{\times N} & \xrightarrow{\rho} & \Gamma T((\mathfrak{g}^*)^{\times N}) \end{array}$$

(here i is the inclusion of $\mathfrak{g}^{\times N}$ in $\mathcal{E}((\mathfrak{g}^*)^{\times N})$ as a set of linear functions and ρ is the Lie algebra homomorphism corresponding to the coadjoint action). It leads to the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\psi^t} & \mathcal{E}(\mathcal{O}) \\ \| & & \downarrow \eta^t(\cdot) \\ \mathfrak{g} & \xrightarrow{\rho^d} & \Gamma T\mathcal{O}, \end{array}$$

where ρ^d is the restriction of ρ to the diagonal, ψ^t is defined as

$$\psi^t(x) = \frac{1}{t_1 + a_1 t_2} p_1^*(x)|_{\mathcal{O}} + \cdots + \frac{1}{t_1 + a_N t_2} p_N^*(x)|_{\mathcal{O}},$$

x in the RHS being understood as a function on $\mathfrak{g}^*$. So assumption (4) of Theorem 4.2 is satisfied, it remains to check assumption (3). This will be done with the help of the commutative diagram

$$\begin{array}{ccc} \mathfrak{g}_j & \xrightarrow{\psi_j} & \mathcal{E}(S_j) \\ \| & & \downarrow \eta^{e_j}(\cdot) \\ \mathfrak{g}_j & \xrightarrow{\rho^d|_{\mathfrak{g}_j}} & \Gamma T S_j. \end{array}$$

Here $\mathfrak{g}_j$ is the Lie algebra of the stabilizer $G_j = G_{\mathfrak{g}^*}^{x_j}$ of a symplectic leaf $S_j = Gx_1 \times \cdots \times Gx_{j-1} \times x_j \times Gx_{j+1} \times \cdots \times Gx_N$, $\rho^d|_{\mathfrak{g}_j}$ is the restriction to $\mathfrak{g}_j$ of the above mentioned map ρ^d, and ψ_j is given by the formula

$$\psi_j(x) = \frac{1}{a_j - a_1} p_1^*(x)|_{S_j} + \cdots + \frac{1}{a_j - a_{j-1}} p_{j-1}^*(x)|_{S_j} + \frac{1}{a_j - a_{j+1}} p_{j+1}^*(x)|_{S_j} + \cdots + \frac{1}{a_j - a_N} p_N^*(x)|_{S_j}, \quad x \in \mathfrak{g}_j \subset \mathcal{E}(\mathfrak{g}^*).$$

Thus, all the assumptions of Theorem 4.2 are checked. In order to finish the proof we need to use condition (2) of equality of ranks from the definition of admissibility. □

Corollary 5.4. *The moment map $\mu_t : \mathcal{O} \to \mathfrak{g}^*$ for the G-action on $(\mathcal{O}, \eta^t)$ is given by the restriction to $\mathcal{O}$ of the following map*:

$$(\mathfrak{g}^*)^{\times N} \ni (x_1, \ldots, x_N) \mapsto \frac{1}{t_1 + a_1 t_2} x_1 + \cdots + \frac{1}{t_1 + a_N t_2} x_N.$$

Proof. Follows from the proof of Theorem 5.3. □

Corollary 5.5. *The set of first integrals $Z^{\{(\eta^t)'\}}$ of the reduced Kronecker bi-Poisson structure coincides with the family of functions*

$$\mathcal{F} = \sum_{t\in\mathbb{C}^2\setminus E} \mu_t^*(Z^{\eta_{\mathrm{can}}}),$$

considered as functions on M/G.

Proof. Follows from Corollary 2.19. See also Corollary 4.3. □

Corollary 5.6. *Assume that G satisfies the condition* $\operatorname{codim}\operatorname{Sing}\mathfrak{g}^* \geq 3$ *of the argument translation method* (*see* Example 3.8). *Then for any fixed $t_0 \in \mathbb{C}^2 \setminus E$ and any regular $a \in \mathfrak{g}^*$ we get a complete involutive set of functions on $\mathcal{O}$*

$$\mathcal{G}^{t_0} = \sum_{t\in\mathbb{C}^2\setminus E} \mu_t^*(Z^{\eta_{\mathrm{can}}}) + \mu_{t_0}^*(Z^{\{\eta^t_{\mathrm{AT}}\}}).$$

Proof. Follows from Corollary 4.3. □

In the remaining part of this section we want to discuss two aspects of applicability of Theorem 5.3: which orbits are admissible and what happens in real case.

Theorem 5.7. *Assume G is semisimple. Then a generic $G^{\times N}$-orbit $\mathcal{O} = Gx_1 \times \cdots \times Gx_N \subset (\mathfrak{g}^*)^{\times N}$ is admissible for any $N \geq 2$.*

Proof. We will first prove condition (2) of Definition 5.2. It follows from the well-known fact (see [1] for example), that the stabilizers of generic elements in the dual space to any Lie algebra are abelian, and from the equality of dimensions: $\operatorname{rank} G = \dim G^{x_1}_{\mathfrak{g}^*} = \cdots = \dim G^{x_N}_{\mathfrak{g}^*}$.

The first condition of the definition of admissibility requires some additional preparations. □

Lemma 5.8. *Let $K \subset G$ be a maximal compact subgroup. Then the principal orbital type stabilizer $K^x_{\mathfrak{g}^*} \subset K$ of an element $x \in \mathfrak{g}^*$ under the coadjoint action of K on $\mathfrak{g}^*$ is at most discrete* (*finite*).

Proof. (The idea of this proof was communicated to the author by Prof. Sam Evens.) For this proof we identify $\mathfrak{g}^*$ and $\mathfrak{g}$ using the Killing form. We claim that the Lie algebra $\mathfrak{k}^x$ of a principal orbital type stabilizer K^x for the K-action on $\mathfrak{g}^*$ is trivial. Indeed, Theorem 3.6 of [13] shows that for any nilpotent element $e \in \mathfrak{g}$ the subalgebra $\mathfrak{g}_e = \operatorname{ad} e(\mathfrak{g}) \cap \mathfrak{g}^e$ consists of nilpotent elements. If, moreover, e is a principal nilpotent element (see [13], Section 5.2) it can be easily seen that $\mathfrak{g}_e = \mathfrak{g}^e$. However, each element of $\mathfrak{k}$ is semisimple; thus $\mathfrak{k}^e = \mathfrak{k} \cap \mathfrak{g}^e = \{0\}$. Of course, this implies the triviality of $\mathfrak{k}^x$. □

Continuation of the proof. Now we are able to complete the proof of Theorem 5.7. Since the K-action on $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{k}$ is diagonal, it follows from the above lemma that for a generic pair $(a, b) \in \mathfrak{k}^* \oplus \mathfrak{k}^*$ the intersection of stabilizers $K^a_{\mathfrak{k}^*} \cap K^b_{\mathfrak{k}^*}$ is finite. The complexification gives the discreteness of the intersection $G^a_{\mathfrak{g}^*} \cap G^b_{\mathfrak{g}^*}$ for a generic pair $(a, b) \in \mathfrak{g}^* \oplus \mathfrak{g}^*$. This implies the result.

Remark 5.9. Theorem 5.7 shows that Theorem 5.3 can be applied to semisimple Lie groups and generic orbits in $(\mathfrak{g}^*)^{\times N}$. We also note that:

(1) Corollary 5.6 is also valid for this data since the condition $\operatorname{codim} \operatorname{Sing} \mathfrak{g}^* \geq 3$ of the argument translation method (see Example 3.8) holds in the semisimple case.
(2) Theorem 5.3 can be also applied for nonsemisimple Lie groups: condition (2) of definition of admissibility Definition 5.2 holds for any Lie algebra $\mathfrak{g}$ and for the stabilizers G^{x_j} of generic points $x_j \in \mathfrak{g}^*$ (see proof of Theorem 5.7); condition (1) should be achieved at least for the algebras with the trivial center by increasing the number of components N.
(3) Another possibility for application of Theorem 5.3 are nongeneric orbits, for example, rank of the stabilizer G^x of any semisimple element $x \in \mathfrak{g}^*$ coincides with rank G for semisimple G (see [4, Chapter 2] for example).

Remark 5.10. Since the complexification of a real semisimple Lie group is complex semisimple, all the results of this section are valid in real setting, i.e. for a real semisimple group G and different $a_1, \dots, a_N \in \mathbb{R}$. All proofs remain the same, only the arguments concerning the proof of condition (1) of the definition of the admissibility for generic orbits require additional considerations.

Proposition 5.11. *Let G be a real semisimple Lie group with the Lie algebra* $\mathfrak{g}$. *Then the generic stabilizer of the G-action on* $(\mathfrak{g}^*)^{\times N}$, $N \geq 2$, *is at most discrete*.

Proof. Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. Then by Lemma 5.8 the set $\mathcal{N}$ of all points $x \in (\mathfrak{g}^{\mathbb{C}})^* \times (\mathfrak{g}^{\mathbb{C}})^*$ with the nontrivial stabilizer $(\mathfrak{g}^{\mathbb{C}})^x$ (with respect to the diagonal action of $\mathfrak{g}^{\mathbb{C}}$) is a proper complex algebraic set. The intersection $\mathcal{N}' = \mathcal{N} \cap \mathfrak{g}^* \times \mathfrak{g}^*$ is a proper real algebraic set, and for $x \in \mathfrak{g}^* \times \mathfrak{g}^* \setminus \mathcal{N}'$ the corresponding real stabilizer $\mathfrak{g}^x = (\mathfrak{g}^{\mathbb{C}})^x \cap \mathfrak{g}$ is trivial. □

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